1 Null hypothesis holds

In this case the true data come from the distribution

$$\mathcal{N}_2(\mu, \Sigma),$$

where

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Further we chose

$$\mathbb{A} = \begin{pmatrix} 1 & 1 \end{pmatrix} \text{ and } a = 0.$$

Hypothesis and alternative: $\mathbf{H}_0 : \mathbb{A}\mu = a$, $\mathbf{H}_1 : \mathbb{A}\mu \neq a$ or

$\mathbf{H}_0 : \mu_1 + \mu_2 = 0$, $\mathbf{H}_1 : \mu_1 + \mu_2 \neq 0$.

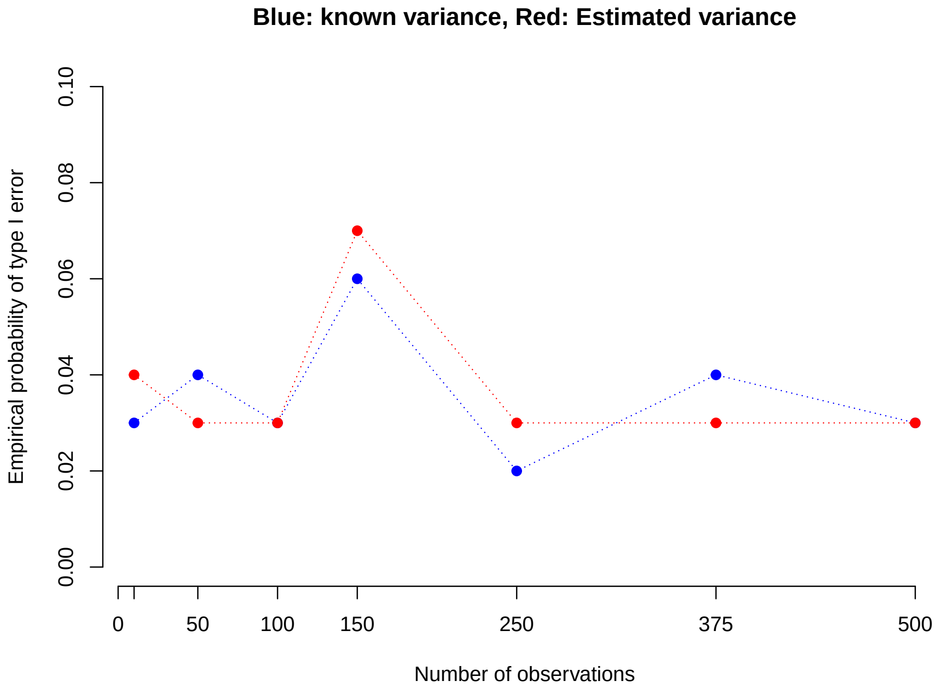
Test statistic with known variance:

$$T_n = n\left(\mathbb{A}\overline{X}_n - a\right)^\top \left(\mathbb{A}\Sigma\mathbb{A}^\top\right)^{-1} \left(\mathbb{A}\overline{X}_n - a\right) \overset{\mathbf{H}_0}{\sim} \chi^2_1.$$

Test statistic with unknown variance:

$$T_n = (n-1)\left(\mathbb{A}\overline{X}_n - a\right)^\top \left(\mathbb{A}\mathcal{S}\mathbb{A}^\top\right)^{-1} \left(\mathbb{A}\overline{X}_n - a\right) \overset{\mathbf{H}_0}{\sim} F_{1,n-1}.$$

We perform the above mentioned tests samples with *sample size* $\in \{10, 50, 100, 250, 375, 500\}$, over larger samples (sample size 1000 and more) the tests had almost same outcomes. Both tests were performed over the same samples in order to prevent random sampling bias. Then we repeat this process for each sample size one hundred times and extract the empirical probabilities of type I error on level $\alpha = 0.05$. We obtain the following plot.



The probability of type I error should be the same for both tests and should be equal to the chosen level α since we control this type of error while designing the tests.

After controling the type I error we try to minimize the type II error, hence the following section should be more interesting.

2 Null hypothesis does not hold

In this case the true data come from the distribution

$$\mathcal{N}_2(\mu, \Sigma),$$

where

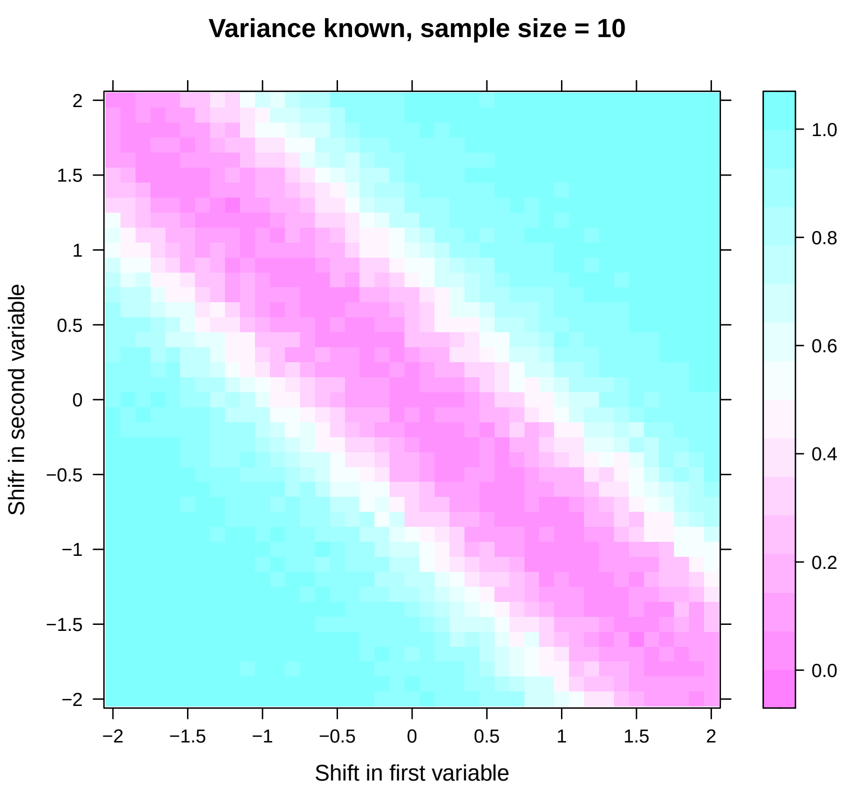
$$\mu = \begin{pmatrix} 0 + x \\ 0 + y \end{pmatrix}, \Sigma = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The matrix $\mathbb{A}$ and value a stay the same

$$\mathbb{A} = \begin{pmatrix} 1 & 1 \end{pmatrix} \text{ and } a = 0.$$

The hypothesis and alternative also stay the same as in section one, however the null hypothesis is violated.

We plot levelplots of empirical probability of type II error for the *sample size* $\in \{10, 50, 500\}$ for both tests and for the shifts in true mean value $x, y \in \{-2, -1.9, -1.8, \ldots, 1.8, 1.9, 2\}$ (special case $x = -y$ on those plot shows type I error).



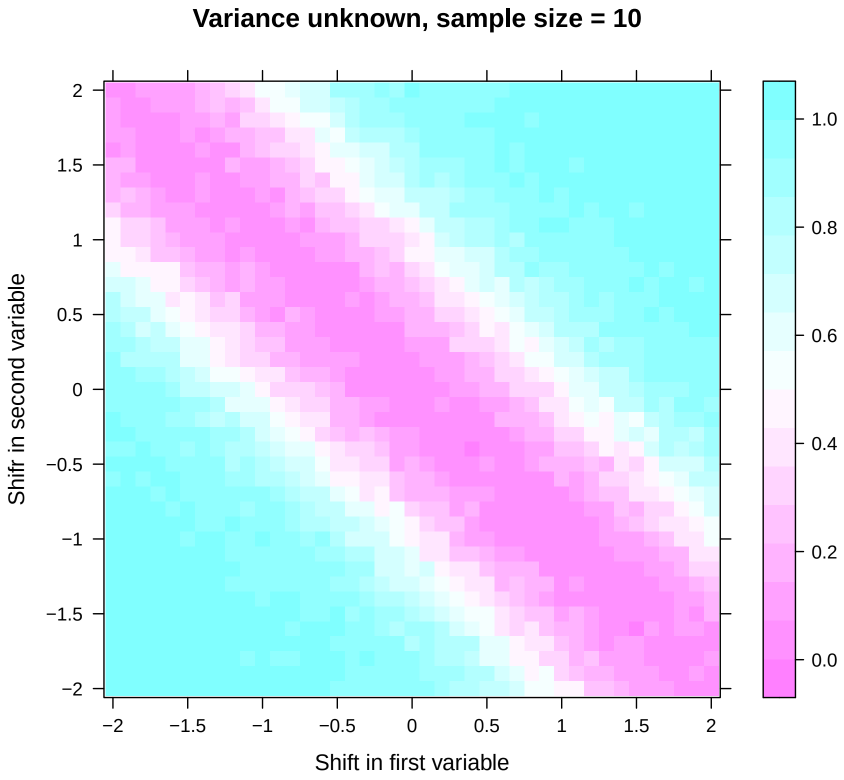


Figure 1: The purple line is little bit thicker for unknown variance than for known variance – bigger probability of type II error for est with unknown variance with sample size = 10.

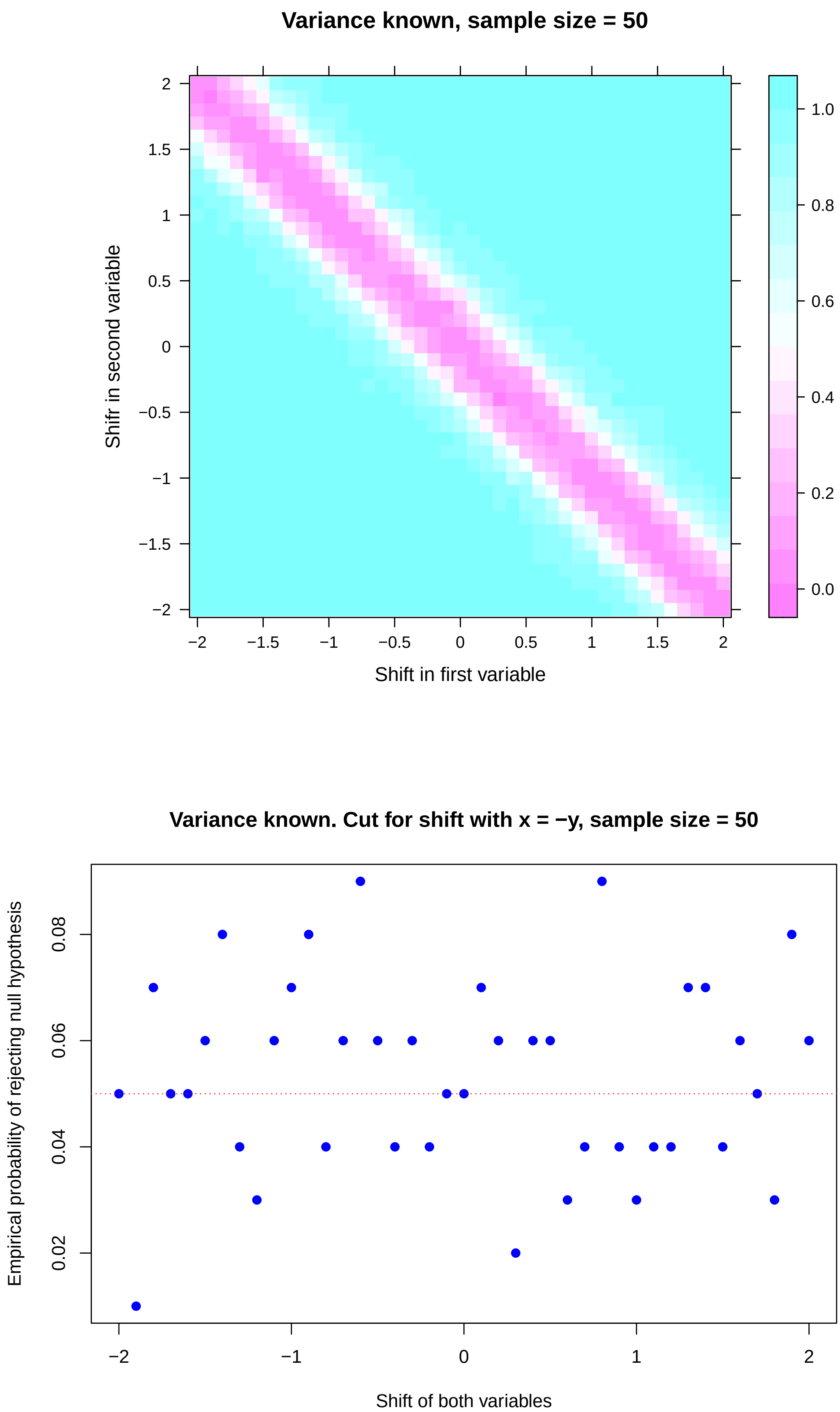


Figure 2: Antidiagonal cross section, empirical type I error.

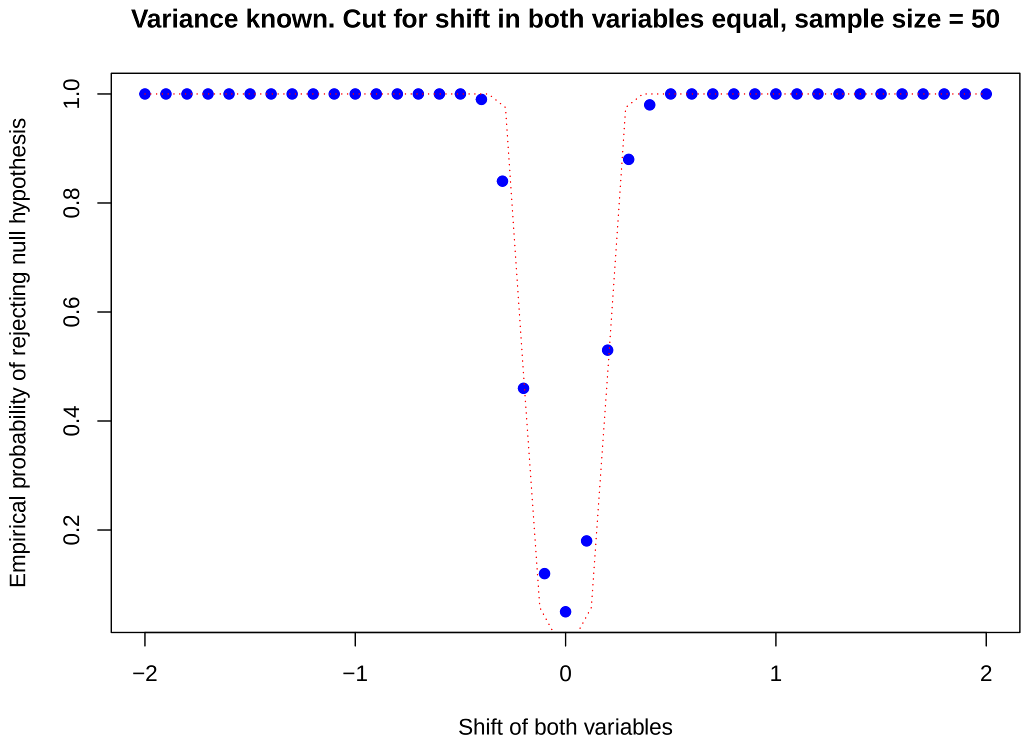
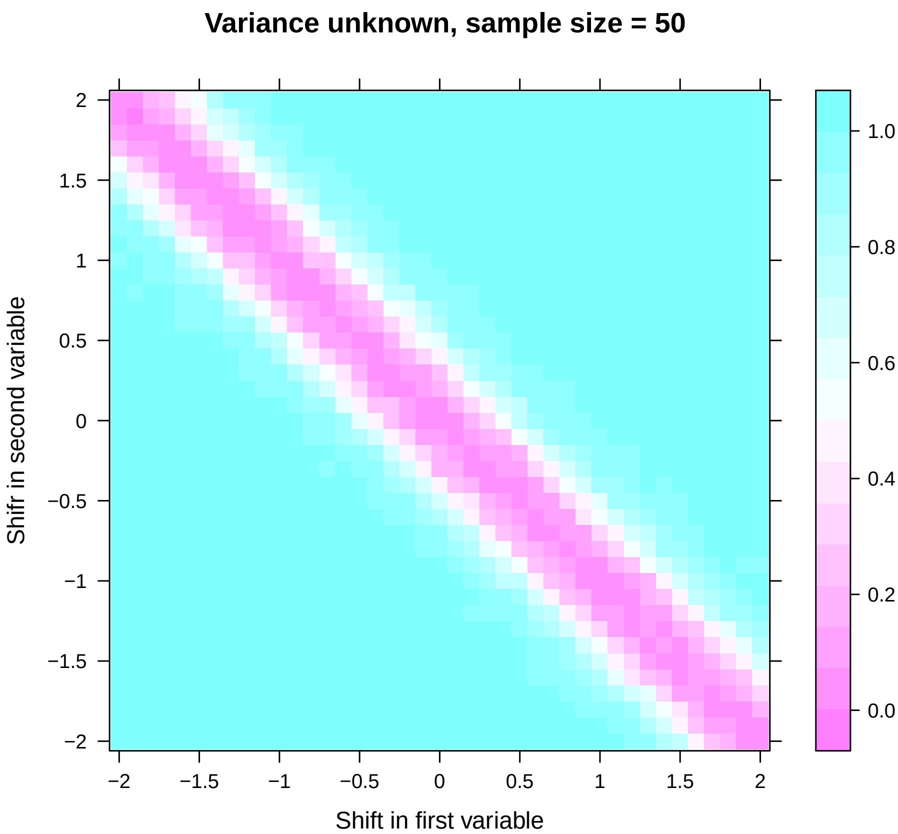


Figure 3: Diagonal cross section, empirical probability of type II error.



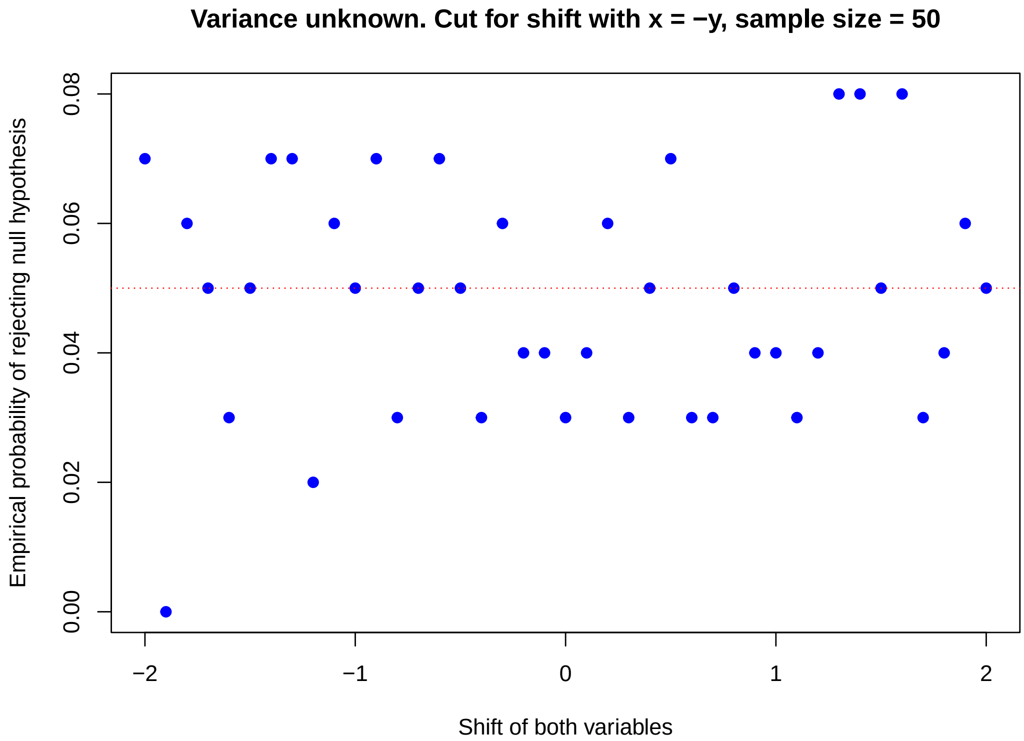


Figure 4: Antidiagonal cross section, empirical type I error.

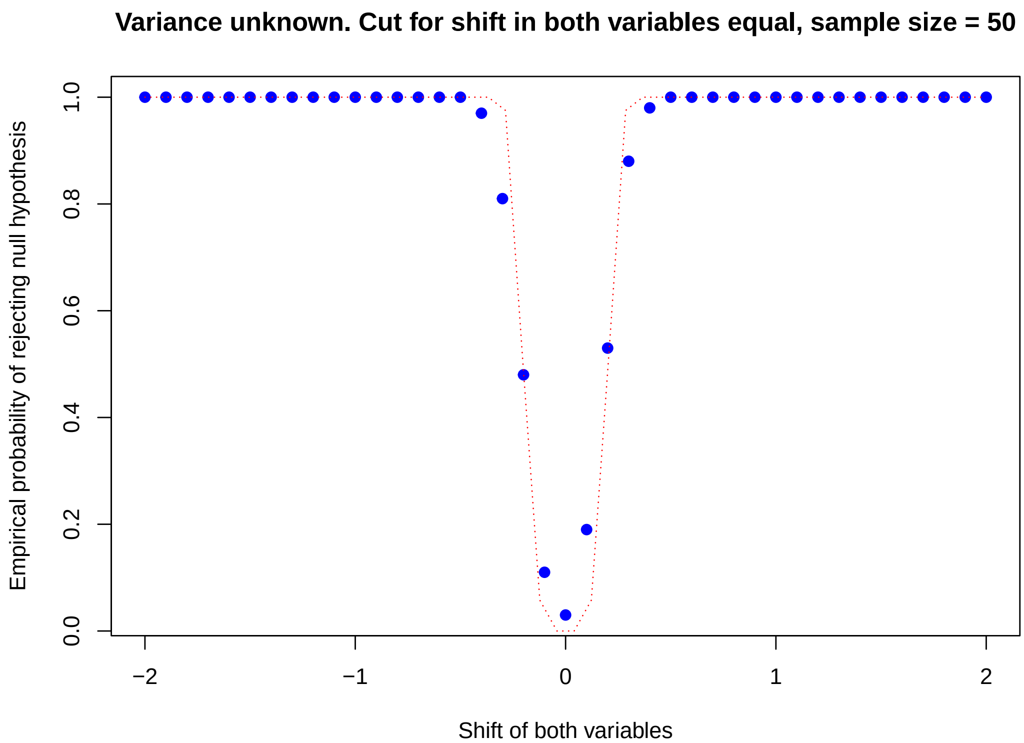
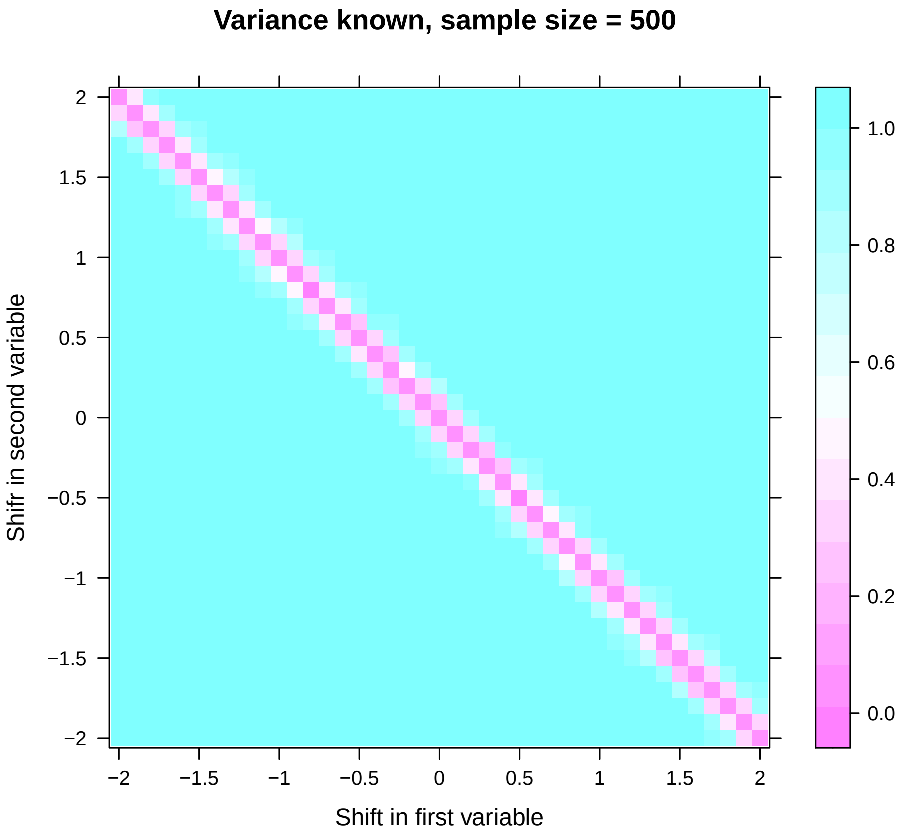
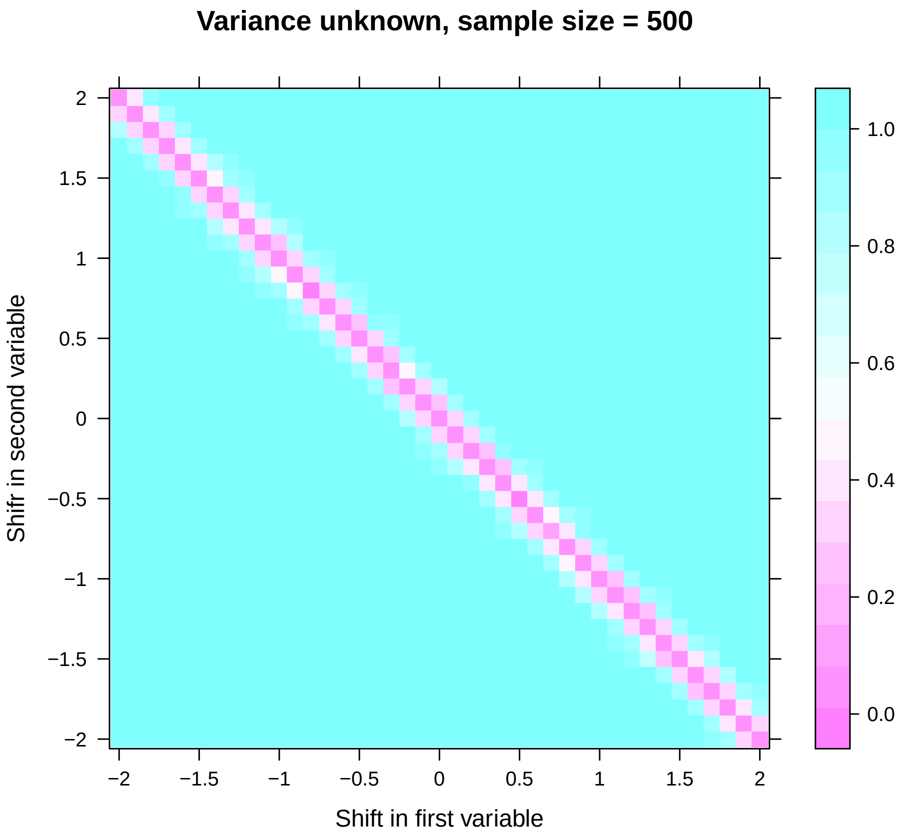


Figure 5: Diagonal cross section, empirical probability of type II error.





The difference between the two tests seems significant only over very small samples.